

RELATIVISTIC PSEUDO GAUSSIAN OSCILLATORS

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The quantum models of a massive scalar particle inside of an open bag generated by a pseudo-Gaussian conformally flat (1+1) metrics are investigated. The potential of a free moving test particle, in the generated metric, has Gaussian asymptotic behavior, approaching to the potential of harmonic oscillator in the limit of zero. The energy levels are calculated using numerical methods, calculations are based on efficient method of generating functionals.

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The non-relativistic harmonic oscillator (HO) is one of the simplest and most useful system in physics. However, its relativistic generalization is not well defined. First attempt were given by Yukawa[1] and further developed by several authors[2]. These attempts were based on covariant generalization $x_\mu x^\mu$ of non relativistic potential. The oscillator based on Dirac equation were also analysed by Moshinsky[3] et al. Other proposals were based on the generalization of the symmetry algebra of quantum operators of quantum free systems [4]. Furthermore, geometric models with different metrics were used to simulate relativistic oscillators [5], [6]. Systems with harmonic oscillator [7] and relativistic harmonic oscillator [8] behaviour can also be induced by a Gaussian potential. Recently a new proposal [9] was made by introducing pseudo-Gaussian potentials to describe a new family of quantum models with HO behavior when approaching to zero. Here we would like to investigate the relativistic case of this recently proposed models, by using pseudo-Gaussian potentials. We have to remark that all models [6][8][9] have the same limit: the nonrelativistic HO. This indicates that this work could be done in the energy basis of HO. This means that the pseudo-Gaussian potential must behave as HO one in some cases. We will see that with suited coefficients this will be accomplished.

The matrix elements of Hamiltonian operator can be calculated in the energy basis of HO. This can be done by using the method of generating functions [10].

Let's consider the quantum model of a scalar test particle with the mass M freely moving inside a pseudo-Gaussian bag, simulated by the (1+1) geometry, defined in a manifold within a local static chart by the line element:

$$ds^2 = g_{00}dt^2 + g_{11}dx^2 = g(x)(dt^2 - dx^2)$$

where

$$g(x) = 1 + v(x) = 1 + \left(\lambda + \sum_{i=1}^r C_i x^{2i} \right) \exp(-\omega^2 x^2) \quad (1)$$

with $\omega > 0$ and λ an arbitrary ground energy parameter, used to establish the depth [9] of the potential well. Taking into consideration some further conditions on the coefficients C_i , they will be introduced latter. Considering an observer in $x = 0$ and since the metric is static the energy is conserved, so the Klein-Gordon equation for the free test particle writes $[\square + M^2]\Phi = 0$ and allows solutions of the form:

$$\Phi^{(+)}(x, t) = \frac{1}{\sqrt{2E}} U_E(x) e^{-iEt}; \quad \Phi^{(-)} = (\Phi^{(+)})^*. \quad (2)$$

Thus it is obtained:

$$\left[-\frac{d^2}{dx^2} + V(x) \right] U_E(x) = (E^2 - M^2) U_E(x) \quad (3)$$

where the potential [9] is:

$$V(x) = M^2 \left(\lambda + \sum_{i=1}^r C_i x^{2i} \right) \exp(-\omega^2 x^2) \quad (4)$$

The free motion in the metric (1) can be seen as one in the potential (4). Thus according to equation (3) the energy spectrum has a discrete part in the domain $[0, M)$ and a continuous one for $E > M$. The eigenfunctions of discrete part must be square integrable,

$$\langle U_{E'} | U_E \rangle = \int dx [U_E(x)]^* U_{E'}(x)$$

The problem of finding the discrete energy levels cannot be solved explicitly in the potential (4). However, we can identify the energy levels step by step up to an arbitrary order, using computational methods. In this way, introducing the variable $\xi = \sqrt{M\omega}x$ and the parameters k respective ν by:

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$$k = \frac{M}{\omega}, \quad 2\nu + \lambda(k-1) + 1 = \frac{E^2 - M^2}{M\omega} \quad (5)$$

the equation (3) becomes:

$$\Delta_{\lambda k}^r U_\nu = (2\nu + 1)U_\nu \quad (6)$$

where U_ν denotes U_E . The operator from the left side of the above equation is:

$$\Delta_{\lambda k}^r = -\frac{d^2}{dx^2} + W_{\lambda k}^r(\xi) - \lambda(k-1) \quad (7)$$

and the potential (4) become:

$$W_{\lambda k}^r(\xi) = k \left(\lambda + \sum_{i=1}^r C_i \xi^{2i} \right) \exp\left(-\frac{\xi^2}{k}\right) - \lambda(k-1) \quad (8)$$

and it is supposed to exhibit Taylor expansions, as follows:

$$W_{\lambda k}^r(\xi) = \xi^2 + \mathcal{O}(\xi^{2r+2}) \quad (9)$$

without terms proportional to $\xi^4, \xi^6, \dots, \xi^{2r}$, thus the potential have a Gaussian asymptotic behavior and in the neighborhood of $\xi = 0$ will behave like HO potential. Furthermore, in the case of nonrelativistic limit, the potential (8) have to be such that the operator (7) becomes the HO operator $\Delta_\lambda = -\frac{d^2}{dx^2} + \xi^2 + \lambda$. These two conditions are accomplished if the coefficients C_i have the following form:

$$C_i := \frac{\lambda + i}{k^i i!} \quad (10)$$

In consequence it is easy to verify that:

$$\lim_{r \rightarrow +\infty} W_{\lambda k}^r(\xi) = \xi^2 + \lambda \Rightarrow \lim_{r \rightarrow +\infty} \Delta_{\lambda k}^r = -\frac{d^2}{dx^2} + \xi^2 + \lambda \quad (11)$$

and

$$\lim_{k \rightarrow +\infty} \Delta_{\lambda k}^r = -\frac{d^2}{dx^2} + \xi^2 + \lambda. \quad (12)$$

The potential (8) together with coefficients (10) is named pseudo-Gaussian potential, changing r and λ a class of pseudo-Gaussian potentials, denoted $(k, \lambda)^r$, are obtained [9]. We have to point out that the coefficients (10) are not uniquely defined, however the coefficients used in [9] do not satisfy condition (12). Taking into consideration the behavior (11) and (12) of the potential it is convenient to work in the energy basis of HO, $\{|n\rangle | n = 0, 1, \dots\}$. In this basis the operator $\Delta_{\lambda k}^r$ could

be put in diagonal form in the desired approximation by fixing $n = N$. A $(N \times N)$ truncated matrix is obtained, thus we can solve the eigenvalue problem (6) by calculating the numeric values of ν which correspond to the discrete energy levels $\epsilon_n = 2\nu_n + 1$. In this parametrization ν is similar to the quantum number of HO and its values verify the condition:

$$2\nu + 1 < -\lambda(k-1) \quad (13)$$

The values for ν are close to integers and will see that the particle behaves like a HO one. To find this values we will use perturbations theory, in the energy basis of HO, using the technique of generating functions.

The generating function, in this case can be written as follows:

$$F_\tau(\xi) = \left(\frac{1}{\pi}\right)^{1/4} \exp\left(-\frac{\xi^2}{2} + 2\xi\tau + \tau^2\right) \quad (14)$$

which yields the HO eigenfunctions normalized in the ξ -scale as:

$$u_n(\xi) = \langle \xi | n \rangle = \frac{1}{\sqrt{(n!2^n)}} \frac{d^n F_\tau(\xi)}{d\tau^n} \Big|_{\tau=0} \quad (15)$$

The matrix elements of any operator \mathcal{X} , in this basis, can be derived from the related generating functional,

$$Z_{\sigma, \tau}[\mathcal{X}] = \int d\xi F_\sigma(\xi) [\mathcal{X} F_\tau](\xi) \quad (16)$$

according with:

$$\langle n | \mathcal{X} | m \rangle = \frac{1}{\sqrt{n!m!2^{n+m}}} \partial_\sigma^n \partial_\tau^m Z_{\sigma, \tau}[\mathcal{X}] \Big|_{\sigma=\tau=0} \quad (17)$$

In general the integral (16) reduces to some known Gaussian integrals. For example in the case of HO: $Z_{\sigma, \tau}[\Delta_\lambda] = (1 + \lambda + 4\sigma, \tau) \exp(2\sigma, \tau)$ with the matrix elements $\langle n | \mathcal{X} | m \rangle = (2n + \lambda + 1) \delta_{nm}$.

The generating functional of an arbitrary model $(\lambda, k)^r$ is given by:

$$Z_{\sigma, \tau}[\Delta_{\lambda k}^r] = Z_{\sigma, \tau}\left[-\frac{d^2}{d\xi^2}\right] + Z_{\sigma, \tau}[W_{\lambda, k}^r - \lambda(k-1)] \quad (18)$$

the terms that can be calculated as Gaussian integrals are:

$$Z_{\sigma, \tau}\left[-\frac{d^2}{d\xi^2}\right] = \left[\frac{1}{2} - (\sigma - \tau)^2\right] \exp(2\sigma\tau), \quad (19)$$

$$Z_{\sigma, \tau}\left[\exp\left(-\frac{\xi^2}{k}\right)\right] = \frac{\sqrt{k}}{\sqrt{k+1}} \exp\left(2\sigma\tau - \frac{1}{\sqrt{k+1}}(\sigma + \tau)^2\right), \quad (20)$$

using the identity $\xi^{2i} e^{-\frac{\xi^2}{k}} = [k^{2i} \partial_k^i] e^{-\frac{\xi^2}{k}}$ to rewrite the generating functional for the potential term, we can conclude:

$$Z_{\sigma,\tau}[\Delta_{\lambda,k}^r] = \left(\frac{1}{2} - (\sigma - \tau)^2 - \lambda(k-1) \right) \exp(2\sigma\tau) \quad (21)$$

$$+ k \left(\lambda + \sum_{i=1}^r C_i [k^{2i} \partial_k^i] \right)$$

$$\times \left\{ \frac{\sqrt{k}}{\sqrt{k+1}} \exp \left(2\sigma\tau - \frac{1}{\sqrt{k+1}} (\sigma + \tau)^2 \right) \right\}$$

The expression (21) for the generating functional will be used to compute the matrix elements of the operator $\Delta_{\lambda,k}^r$. The methodology used will be described below as follows:

A concrete value for r have to be taken. The value for r determine the shape of the potential (8). A value $r = 3$ have been considered. By fixing λ , we have a concrete model $(k, \lambda)^r$ i.e. the shape of the potential (8) is completely determined. To compute the eigenvalues ϵ_n we will truncate the infinite dimensional matrix of the operator $\Delta_{\lambda,k}^r$ to the $(N \times N)$ block. Thus, the matrix elements, $\langle n | \mathcal{X} | m \rangle$ can be derived one by one using the relation (17) with the calculated generating functional (21).

For each k , N values of eigenvalues $\epsilon_n = 2\nu_n + 1$ will be obtained; where $n = 1, 2, \dots, N$. According to (13) there are a finite number of values of ν_n which could determine discrete energy levels, the other ones overlap the continuous spectrum. Let us denote by D , the largest value for the discrete part, $\nu_1, \nu_2, \dots, \nu_D$ with $\epsilon_n \leq -\lambda(k-1)$, equivalently we can write $E_n \in [0, M)$ for $n = \{1, 2, \dots, D\}$. In what follows we will denote the higher energy from the discrete spectra with ϵ_d .

Calculations have been made, using symbolic and numeric computation. Thus we have analyzed the case with $N = 50$, calculating the dependence of ν_n versus n for the following values of $k = \{11, 21, 41, 61\}$ and with different values of $\lambda = \{-1, -7, -10\}$.

In the figure (1) we have represented the dependence of pseudoquantum number ν_n versus HO quantum number n , in the case of $\lambda = -1$. In this case the upper limit for the eigenvalues is $\epsilon_n \leq k-1$ and represent the separation limit between discrete and continuous spectra. A inflection point is observed near this limit. This means that the distance between energy levels is lowering and is minimum at the separation level. The results are similar in this case of $\lambda = -1$, with those obtained in [8] where $\epsilon_n \leq k$.

The eigenvalues ranging into the discrete domain $\epsilon_d = 2\nu_d + 1$ with $d = \{1, 2, \dots, D\}$ are represented in figure (2).

The values of energy are under the values of HO, but are very close to these, at least for the low energy levels. A closer look at the disposal of the energies it is

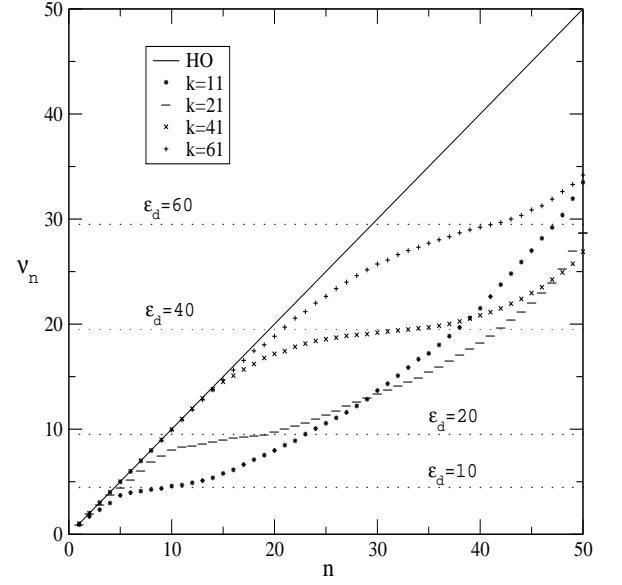


FIG. 1: The dependence of pseudoquantum number ν_n on HO quantum number n with different values for $k = \{11, 21, 41, 61\}$ and $\lambda = -1$ in the order $r = 3$ using a matrix with $N = 50$.

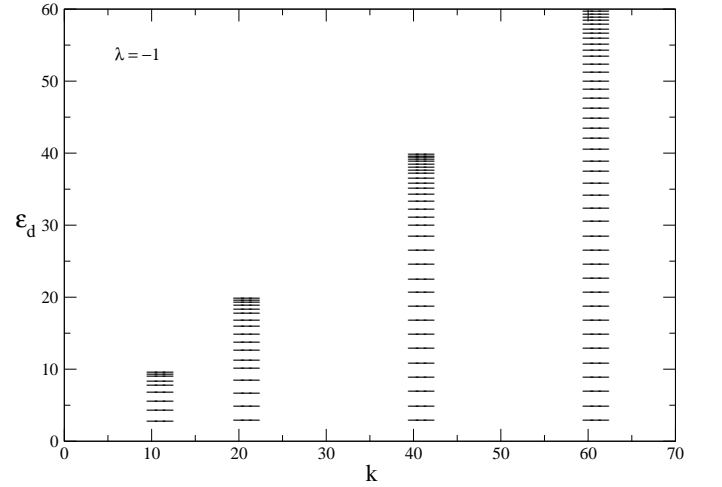


FIG. 2: The dependence of eigenvalues ϵ_n with k in the order $r = 3$ using a matrix with $N = 50$.

shown in the figure (3) the amount of calculated energy are compared with those of HO.

As one can expect increasing the λ , the energy levels are closer to HO ones, so the system can be assimilate with more accuracy with an HO system. This seems to be natural because increasing the absolute value of λ the shape of potential (8) approaching to HO ones. The dependence of ν_n versus n for $\lambda = \{-7, -10\}$ is presented in figure (4).

The number of energy levels is increasing with λ , the distance between levels remaining equidistant, less in the

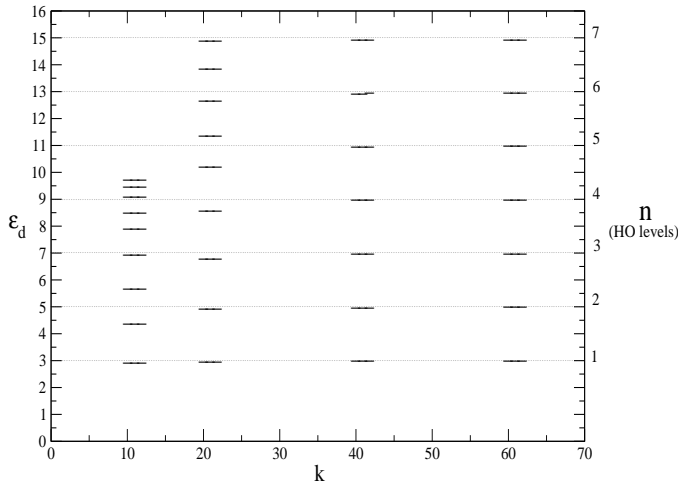


FIG. 3: The comparison between calculated energy ϵ_d and HO energy level $n = \{1, 2, \dots\}$

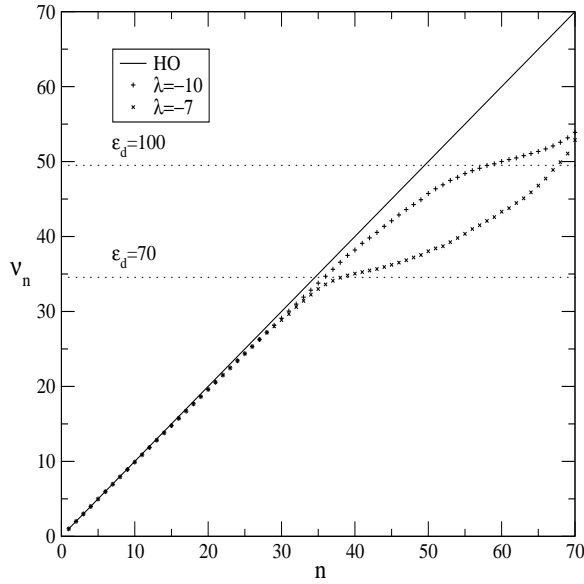


FIG. 4: The dependence of pseudoquantum number ν_n on HO quantum number n with $\lambda = \{-7, -10\}$ and $k = 10$ in the order $r = 3$ using a matrix with $N = 70$.

vicinity of separation point. The eigenvalues ranging into the discrete domain $\epsilon_d = 2\nu_d + 1$ with $d = \{1, 2, \dots, D\}$ are represented in figure (5).

We have shown that a pseudo Gaussian bag with a

massive quantum particle inside, have an energy spectrum with a finite part $E_n \in [0, M)$ and a continuous one $E_n \in [M, \infty)$. The energy levels of the discrete spectra have been calculated in terms of eigenvalues ϵ_d , $d \in \{0, \dots, D\}$ with the relation:

$$E_d = [M^2 + M\omega(\epsilon_d + \lambda(\frac{M}{\omega} - 1))]^{\frac{1}{2}} \quad (22)$$

with a ground energy parameter $\lambda \leq -1$ and $M > \omega$.

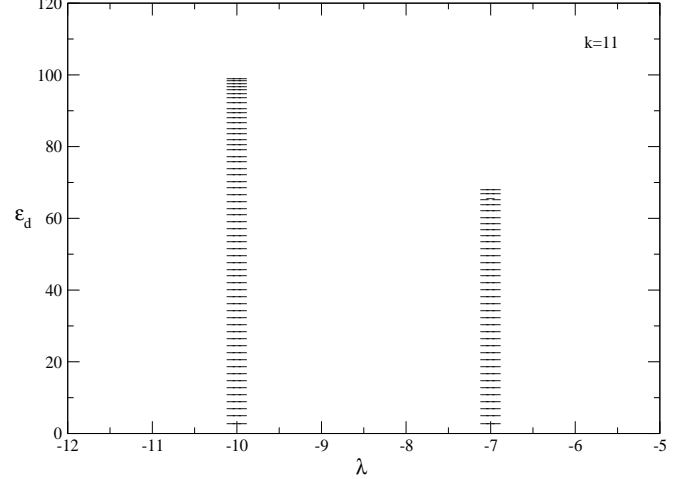


FIG. 5: The dependence of eigenvalues ϵ_n with λ in the order $r = 3$ using a matrix with $N = 70$.

The eigenvalues ϵ_n were calculated numerically, using the method of generating functional.

The model is interesting from physical point of view, because it offers the possibility to investigate small potential wells with arbitrary energy levels from where the quantum particle can escape when a transition from discrete to continuous spectrum becomes possible. Starting with $\lambda = -1$ and $k = 1$ where no energy levels appears, any model $(\lambda, k)^r$ can be obtained with an arbitrary energy levels. We think this model can be used to calculate electronic transitions in semiconductors.

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